# Inertia in Optimization: Acceleration and Adaptivity

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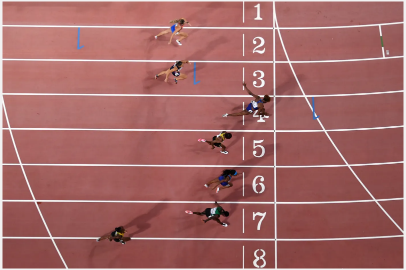
# Framework and motivations

## Optimization, what is this?

$\rightarrow$ Find a set of parameters that minimizes a quantity.

Find the route that minimizes journey time.



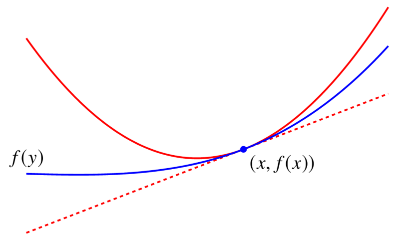

Find the training that leads to the best 100-meter time.

## Minimization problem

$$\min_{x\in\mathbb{R}^N} F(x) = f(x) + h(x),$$

where:

- $f$ is a convex differentiable function having a $L$-Lipschitz gradient,




- $h$ is a convex proper lower semicontinuous function,
- $F$ has a non-empty set of minimizers $X^*$.

## Motivations

$$\min_{x \in \mathbb{R}^N} F(x),$$

Which algorithm is the most efficient according to the **assumptions** satisfied by $F$ and the **expected accuracy**?

$\rightarrow$ **Convergence analysis** of the numerical schemes:

How fast does $F(x_k) - F^*$ decreases?

# Outline

**A classical algorithm: the proximal gradient method** (Combettes and Wajs, '05)

$$\forall k > 0,\ x_k = \text{prox}_{sh}\left(x_{k-1} - s\nabla f(x_{k-1})\right).$$

Composite version of the **Gradient Descent method**:

$$\forall k > 0,\ x_k = x_{k-1} - s\nabla F(x_{k-1}).$$

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Composite version of the **Gradient Descent method**:

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## Convergence guarantees

If $F$ is convex and $s$ is sufficiently small:

$$F(x_k) - F^* = \mathcal{O}\left(k^{-1}\right)$$

$\rightarrow$ Simple but slow!

# Inertia in optimization

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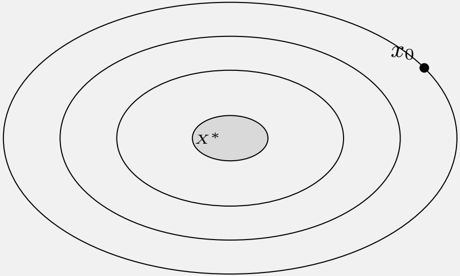
## Illustration



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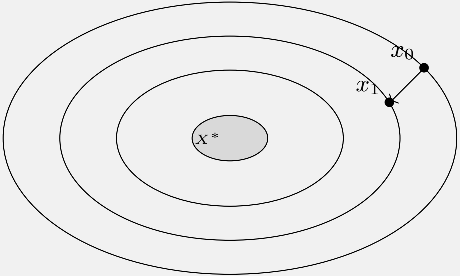

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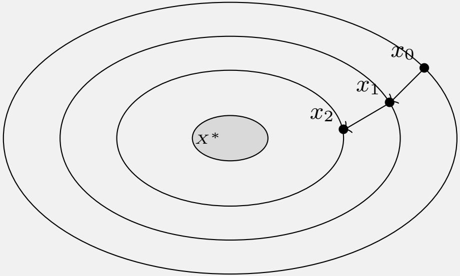

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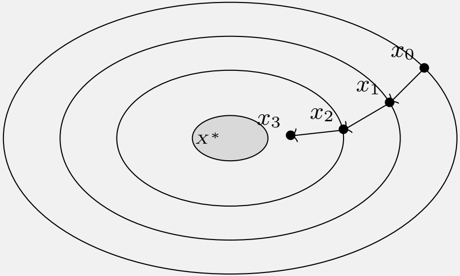

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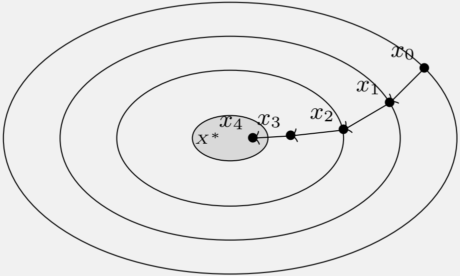

# Inertia



## Introducing inertia

$\rightarrow$ Apply the same transformation to a shifted point.

$$\forall k > 0, \begin{cases} x_k = \text{prox}_{sh}\left(y_{k-1} - s\nabla f(y_{k-1})\right), \\ y_k = x_k + \alpha_k(x_k - x_{k-1}), \end{cases}$$

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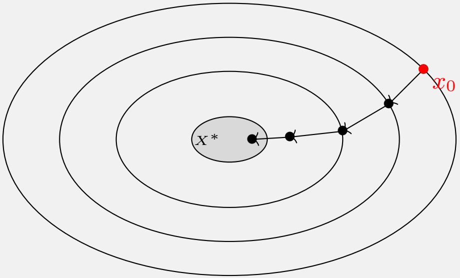

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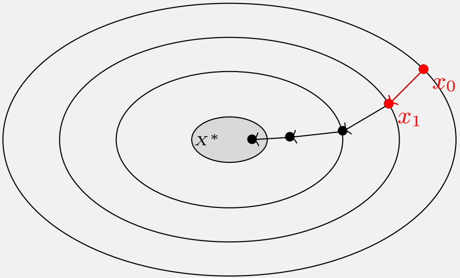

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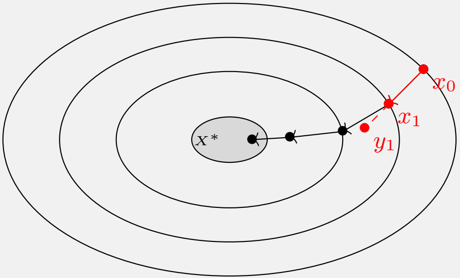

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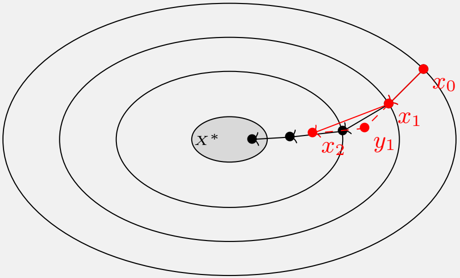

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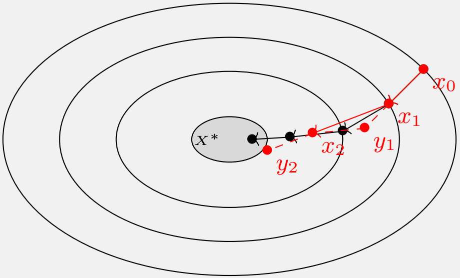

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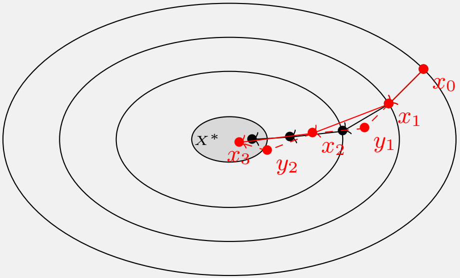

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## Rising question

**How to chose $\alpha_k$?**

# Inertia



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## Rising question

**How to chose $\alpha_k$?**

- **Heavy-Ball schemes** (Polyak,'64, Nesterov,'03, ...): constant friction → $\alpha_k = \alpha$.
- **FISTA** (Beck and Teboulle,'09, Nesterov,'83): vanishing friction → $\alpha_k = \frac{k-1}{k+\alpha-1}$.

# Geometry of convex functions

## Strong convexity ($\mathcal{SC}_\mu$)

$F$ is $\mu$-strongly convex if for all $x \in \mathbb{R}^N$, $g : x \mapsto F(x) - \frac{\mu}{2}\|x\|^2$ is convex.

## Convergence rate of $F(x_k) - F^*$

| **Algorithm** | **Convex** | $\mathcal{SC}_\mu$ |
|---|---|---|
| Proximal gradient method | $\mathcal{O}\left(k^{-1}\right)$ | $\mathcal{O}\left(e^{-\frac{\mu}{L}k}\right)$ |
| Heavy-Ball (constant friction) | $\mathcal{O}\left(k^{-1}\right)$ | $\mathcal{O}\left(e^{-2\sqrt{\frac{\mu}{L}}k}\right)$ |
| FISTA (vanishing friction) | $\mathcal{O}\left(k^{-2}\right)$ | $\mathcal{O}\left(k^{-\frac{2\alpha}{3}}\right)$ |

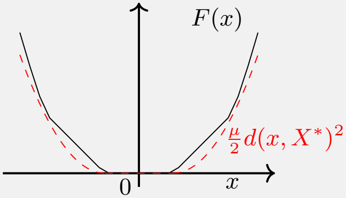
## Classical geometry assumptions

- **Quadratic growth condition ($\mathcal{G}^2_\mu$)**:
  $F$ has a quadratic growth around its set of minimizers if

$$\exists \mu > 0,\ \forall x \in \mathbb{R}^N,\ \frac{\mu}{2} d(x, X^*)^2 \leqslant F(x) - F^*.$$

**Practical example:** LASSO problem:

$$F(x) = \frac{1}{2}\|Ax - y\|^2 + \lambda \|x\|_1.$$



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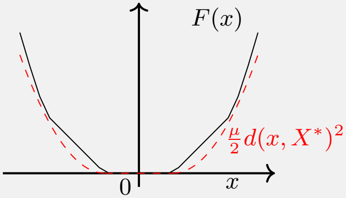


- **Hölderian error bound ($\mathcal{H}^\gamma$)**:
  $F$ has a $\gamma$-Hölderian growth around its set of minimizers (with $\gamma > 2$) if

$$\exists K > 0,\ \forall x \in \mathbb{R}^N,\ K d(x, X^*)^\gamma \leqslant F(x) - F^*.$$

# Outline

## Framework

$$\min_{x\in\mathbb{R}^N} F(x) \text{ for } F \text{ satisfying some geometry assumption.}$$

## What did we know?

| **Algorithm** | $\mathcal{SC}_\mu$ | $\mathcal{G}^2_\mu$ | $\mathcal{H}^\gamma$ | Convexity |
|---|---|---|---|---|
| PGD | $e^{-\frac{\mu}{L}k}$ | | | $k^{-1}$ |
| Heavy-Ball | $e^{-2\sqrt{\frac{\mu}{L}}k}$ | | | $k^{-1}$ |
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# Inertia between convexity and strong convexity



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**If $F$ has a unique minimizer!!**

*in the continuous setting (Begout et al., '15).

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**If $F$ has a unique minimizer!!**

**Is it really necessary?**

*in the continuous setting (Begout et al., ’15).

**How to avoid the uniqueness assumption?**

## Our strategy

Consider **V-FISTA** (Beck,'17, Nesterov,'03):

$$\forall k > 0, \begin{cases} x_k = \text{prox}_{sh}(y_{k-1} - s\nabla f(y_{k-1})) \\ y_k = x_k + \alpha(x_k - x_{k-1}) \end{cases}$$

where $F = f + h$ is such that $\frac{\mu}{2} d(x, X^*)^2 \leqslant F(x) - F^*$ for any $x \in \mathbb{R}^N$.

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where $F = f + h$ is such that $\frac{\mu}{2}d(x, X^*)^2 \leqslant F(x) - F^*$ for any $x \in \mathbb{R}^N$.
**Classical discrete Lyapunov energy for this system:**

$$\mathcal{E}_k = s(F(x_k) - F^*) + \frac{1}{2}\|\lambda(x_k - x^*) + x_k - x_{k-1}\|^2$$

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where $x_k^*$ is the projection of $x_k$ onto the set of minimizers of $F$ denoted $X^*$.

$\rightarrow$ Trickier calculations
$\rightarrow$ No assumption on $X^*$ required!

## Main results: V-FISTA

$$\forall k > 0, \begin{cases} x_k = \text{prox}_{sh}(y_{k-1} - s\nabla f(y_{k-1})) \\ y_k = x_k + \alpha(x_k - x_{k-1}) \end{cases}$$

**Theorem** (Aujol, Dossal, L., Rondepierre,'24): If $F$ satisfies $\mathcal{G}^2_\mu$ , $s = \frac{1}{L}$ and $\alpha = 1 - \frac{5}{3\sqrt{3}}\sqrt{\frac{\mu}{L}}$ :

$$F(x_k) - F^* = \mathcal{O}\left(e^{-\frac{2}{3\sqrt{3}}\sqrt{\frac{\mu}{L}}k}\right)$$

- Uniqueness of the minimizer is not required.
- Theoretical guarantees for non optimal values of $\alpha$.
- Better worst-case bound than any FISTA restart scheme: $\mathcal{O}\left(e^{-\frac{1}{e}\sqrt{\frac{\mu}{L}}k}\right)$.
- $\alpha$ depends on $\frac{\mu}{L}$!

## Main results: FISTA for $\mathcal{G}^2_\mu$

$$\forall k > 0, \begin{cases} x_k = \text{prox}_{sh}(y_{k-1} - s\nabla f(y_{k-1})) \\ y_k = x_k + \dfrac{k-1}{k+\alpha-1}(x_k - x_{k-1}) \end{cases}$$

**Theorem** (Aujol, Dossal, L., Rondepierre,'24): If $F$ satisfies $\mathcal{G}^2_\mu$ , $s = \frac{1}{L}$ and $\alpha \geqslant 3 + \frac{3}{\sqrt{2}}$ :

$$F(x_k) - F^* = \mathcal{O}\left(k^{-\frac{2\alpha}{3}}\right)$$

- Uniqueness of the minimizer is not required.
- Finite time bound available.
- $\alpha$ can be parametrized according to the expected accuracy to get improved performance.

## Main results: FISTA for $\mathcal{H}^\gamma$

$$\forall k > 0, \begin{cases} x_k = \text{prox}_{sh}(y_{k-1} - s\nabla f(y_{k-1})) \\ y_k = x_k + \dfrac{k-1}{k+\alpha-1}(x_k - x_{k-1}) \end{cases}$$

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$$\forall x \in B(x^*, \varepsilon),\ Kd(x, X^*)^\gamma \leqslant F(x) - F^*,$$

then for $\alpha > 5 + \frac{8}{\gamma-2}$:

$$F(x_k) - F^* = \mathcal{O}\left(k^{-\frac{2\gamma}{\gamma-2}}\right) \text{ and } \|x_k - x_{k-1}\| = \mathcal{O}\left(k^{-\frac{\gamma}{\gamma-2}}\right)$$

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**Corollary**: Under the same assumptions, for any $\alpha > 5$, the sequence $(x_k)_{k\in\mathbb{N}}$ converges **strongly** to a minimizer of $F$.

# Inertia between convexity and strong convexity



## What do we know now?

| **Algorithm** | $\mathcal{SC}_\mu$ | $\mathcal{G}^2_\mu$ | $\mathcal{H}^\gamma$ | Convexity |
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## Take-away message

Inertia is **not impacted** by the non uniqueness of the minimizers.

| | $\mathcal{SC}_\mu$ | $\mathcal{G}^2_\mu$ | $\mathcal{H}^\gamma$ | Convexity |
|---|---|---|---|---|
| Best option | HB | HB | FISTA | FISTA |

Jean-François Aujol, Charles Dossal, Hippolyte Labarrière, Aude Rondepierre. Heavy Ball Momentum for Non-Strongly Convex Optimization, 2024, arXiv preprint arXiv:2403.06930.

Jean-François Aujol, Charles Dossal, Hippolyte Labarrière, Aude Rondepierre. Strong Convergence of FISTA Iterates under Hölderian and Quadratic Growth Conditions, 2024, arxiv:2407.17063.

# Outline

# Restart strategies



## Framework

$$\min_{x\in\mathbb{R}^N} F(x),$$

where $F$ satisfies a growth condition ($\mathcal{SC}_\mu$ or $\mathcal{G}^2_\mu$) and the growth parameter $\mu$ is not known.

## First-order methods

In this setting:

- **proximal gradient method**: $F(x_k) - F^* = \mathcal{O}\left(e^{-\frac{\mu}{L}k}\right)$,
- **Heavy-Ball methods**: $F(x_k) - F^* = \mathcal{O}\left(e^{-K\sqrt{\frac{\mu}{L}}k}\right)$ if $\mu$ is known,
- **FISTA** (Beck and Teboulle,'09, Nesterov,'83):

$$\forall k > 0, \begin{cases} x_k = \text{prox}_{sh}(y_{k-1} - s\nabla f(y_{k-1})), \\ y_k = x_k + \dfrac{k-1}{k+2}(x_k - x_{k-1}) \end{cases}$$

$$\rightarrow F(x_k) - F^* = \mathcal{O}\left(k^{-2}\right)$$

# Restart strategies

## Restarting FISTA, why?

- to take advantage of inertia,
- to avoid oscillations.

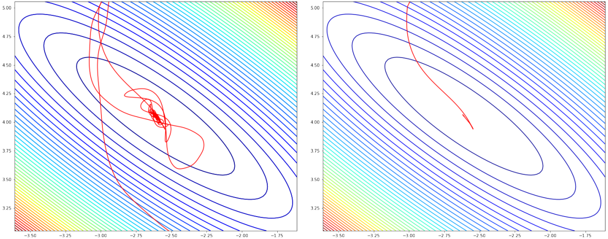

Figure: Projection of the trajectory of the iterates of FISTA (left) and FISTA restart (right) for a least-squares problem ($N = 20$).

# Restart strategies

## Restarting FISTA, how?

**Algorithm 1** : FISTA restart

**Require:** $x_0 \in \mathbb{R}^N,\ y_0 = x_0,\ k = 0,\ i = 0.$
  **repeat**
    $k = k + 1, i = i + 1$
    $x_k = \text{prox}_{sh}(y_{k-1} - s\nabla f(y_{k-1}))$
    **if** Restart condition is $True$ **then**
      $i = 1$
    **end if**
    $y_k = x_k + \frac{i-1}{i+2}(x_k - x_{k-1})$
  **until** Exit condition is $True$

$\rightarrow$ Cutting inertia is equivalent to restarting the algorithm from the last iterate.

# Restart strategies



**Empiric FISTA restart** (O'Donoghue and Candès, '15, Beck and Teboulle, '09)

Restart under some exit condition

- on $F$:

$$F(x_k) > F(x_{k-1}),$$

- on $\nabla F$:

$$\langle \nabla F(y_k), x_k - x_{k-1} \rangle > 0.$$

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- on $\nabla F$:

$$\langle \nabla F(y_k), x_k - x_{k-1} \rangle > 0.$$

**Fixed FISTA restart** (Nesterov, '13, O'Donoghue and Candès, '15...)

Restart every $k^*$ iterations where $k^*$ is defined according to the growth parameter $\mu$. If $k^* = \left\lfloor 2e\sqrt{\frac{L}{\mu}} \right\rfloor$:

$$F(x_k) - F^* = \mathcal{O}\left(e^{-\frac{1}{e}\sqrt{\frac{\mu}{L}}k}\right).$$

Generalization: Scheduled restarts, Roulet and D'Aspremont '17.

## Adaptive FISTA restart

Restart according to the geometry of $F$ and previous iterations.

- Fercoq and Qu, '19: $F(x_k) - F^* = o\left(\exp\left(-\frac{\sqrt{2}-1}{2\sqrt{e}\left(2-\sqrt{\frac{\mu}{\mu_0}}\right)}\sqrt{\frac{\mu}{L}}k\right)\right)$.
- Alamo et al., '19: $F(x_k) - F^* = \mathcal{O}\left(e^{-\frac{1}{16}\sqrt{\frac{\mu}{L}}k}\right)$.
- Alamo et al., '22: $F(x_k) - F^* = \mathcal{O}\left(e^{-\frac{\ln(15)}{4e}\sqrt{\frac{\mu}{L}}k}\right)$, where $\frac{\ln(15)}{4e} \approx \frac{1}{4}$.
- Renegar and Grimmer, '22: $F(x_k) - F^* = \mathcal{O}\left(e^{-\frac{1}{2\sqrt{2}}\sqrt{\frac{\mu}{L}}k}\right)$.

**Introduction of an automatic restart scheme** (Aujol, Dossal, L., Rondepierre,'21)

**Features**: a restart condition that

- does not require to know the growth parameter $\mu$,
- ensures a fast convergence of the method: $F(x_k) - F^* = \mathcal{O}(e^{-\frac{1}{12}\sqrt{\frac{\mu}{L}}k})$,
- is not computationnaly expensive,
- is easy to implement.

**Strategy**

- to estimate $\mu$ at each restart,
- to adapt the number of iterations of the following restart according to this estimation.

Jean-François Aujol, Charles Dossal, Hippolyte Labarrière, Aude Rondepierre. FISTA restart using an automatic estimation of the growth parameter, 2021, ⟨hal-03153525v4⟩.

# Restart strategies

**Algorithm 2** : Automatic FISTA restart

**Require:** $r_0 \in \mathbb{R}^N,\ j = 1,\ C = 6.38.$

$n_0 = \lfloor 2C \rfloor$

$r_1 = \text{FISTA}(r_0, n_0)$

$n_1 = \lfloor 2C \rfloor$

**repeat**

$j = j + 1$

$r_j = \text{FISTA}(r_{j-1}, n_{j-1})$

$\tilde{\mu}_j = \min_{\substack{i \in \mathbb{N}^* \\ i < j}} \frac{4L}{(n_{i-1}+1)^2} \frac{F(r_{i-1}) - F(r_j)}{F(r_i) - F(r_j)}$ Estimation of the parameter $\mu$.

**if** $n_{j-1} \leqslant C\sqrt{\frac{L}{\tilde{\mu}_j}}$ **then**

$n_j = 2n_{j-1}$ Update of the number of iterations per restart.

**end if**

**until** *exit condition is satisfied*

# Restart strategies



## Summary:

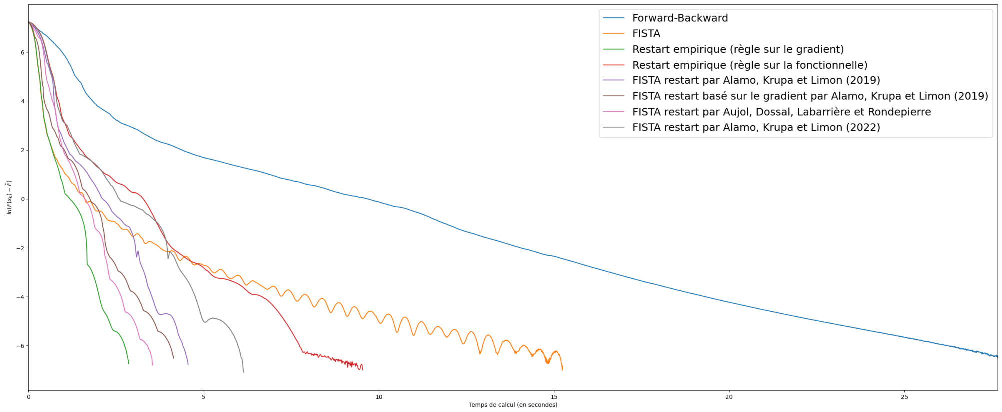
| Algorithm | Convergence rate |
|---|---|
| Forward-Backward | $\mathcal{O}\left(e^{-\frac{\mu}{L}k}\right)$ |
| V-FISTA | $\mathcal{O}\left(e^{-\frac{9}{20}\sqrt{\frac{\mu}{L}}k}\right)$ |
| Optimal FISTA restart | $\mathcal{O}\left(e^{-\frac{1}{e}\sqrt{\frac{\mu}{L}}k}\right)$ |
| Empirical FISTA restart | $\mathcal{O}(k^{-2})$ |
| Fercoq and Qu '19 | $\mathcal{O}\left(e^{-\frac{\sqrt{2}-1}{2\sqrt{e}(2-\sqrt{\frac{\mu}{\mu_0}})}\sqrt{\frac{\mu}{L}}k}\right)$ |
| Alamo et al. '19 | $\mathcal{O}\left(e^{-\frac{1}{16}\sqrt{\frac{\mu}{L}}k}\right)$ |
| Alamo et al. '22 | $\mathcal{O}\left(e^{-\frac{\ln(15)}{4e}\sqrt{\frac{\mu}{L}}k}\right)$ |
| Renegar and Grimmer '22 | $\mathcal{O}\left(e^{-\frac{1}{2\sqrt{2}}\sqrt{\frac{\mu}{L}}k}\right)$ |
| **Automatic FISTA restart** | $\mathcal{O}\left(e^{-\frac{1}{12}\sqrt{\frac{\mu}{L}}k}\right)$ |

## Image inpainting:

$$\min_x F(x) := \frac{1}{2}\|Mx - y\|^2 + \lambda\|Tx\|_1,$$

where $M$ is a mask operator and $T$ is an orthogonal transformation ensuring that $Tx^0$ is sparse.

## Image inpainting:



**What if the Lipschitz constant $L$ is not known?**

## Combining backtracking and restarting: Free-FISTA (Aujol, Calatroni, Dossal, L., Rondepierre, '24)

By combining a **backtracking strategy** and a **restarting strategy**, Free-FISTA automatically estimates $\mu$ and $L$.

- Still efficient if $L$ **is not known**.
- Adaptation to the **local geometry of** $F$.
- **Convergence guarantees**: $F(x_k) - F^* = \mathcal{O}\left(e^{-\frac{\sqrt{\rho}}{12}\sqrt{\frac{\mu}{L}}k}\right)$.

Jean-François Aujol, Luca Calatroni, Charles Dossal, Hippolyte Labarrière, Aude Rondepierre. Parameter-Free FISTA by Adaptive Restart and Backtracking, 2024, *SIAM Journal on Optimization*.

# An other approach



**FISTA is far from optimal for functions satisfying strong growth conditions!**

## Recall

| Algorithm | $\mathcal{SC}_\mu$ | $\mathcal{G}^2_\mu$ |
|---|---|---|
| FISTA | $k^{-\frac{2\alpha}{3}}$ | $k^{-\frac{2\alpha}{3}}$ |
| Optimal FISTA restart | $e^{-\frac{1}{e}\sqrt{\frac{\mu}{L}}k}$ | $e^{-\frac{1}{e}\sqrt{\frac{\mu}{L}}k}$ |
| | | |

# An other approach



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## Recall

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| Optimal FISTA restart | $e^{-\frac{1}{e}\sqrt{\frac{\mu}{L}}k}$ | $e^{-\frac{1}{e}\sqrt{\frac{\mu}{L}}k}$ |
| V-FISTA (HB) | $e^{-\sqrt{\frac{\mu}{L}}k}$ | $e^{-\frac{2}{3\sqrt{3}}\sqrt{\frac{\mu}{L}}k}$ |

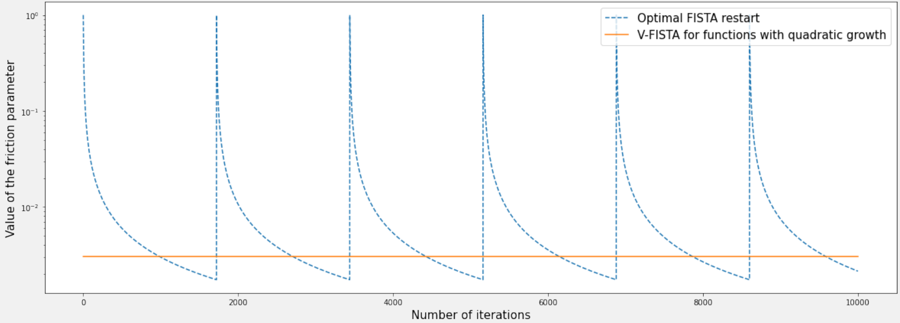
# An other approach



## Behavior of the friction parameter

$$\forall k > 0, \begin{cases} x_k = \text{prox}_{sh}\left(y_{k-1} - s\nabla f(y_{k-1})\right), \\ y_k = x_k + \alpha_k (x_k - x_{k-1}), \end{cases}$$

$\rightarrow$**Friction parameter:** $1 - \alpha_k$



## Behavior of the friction parameter

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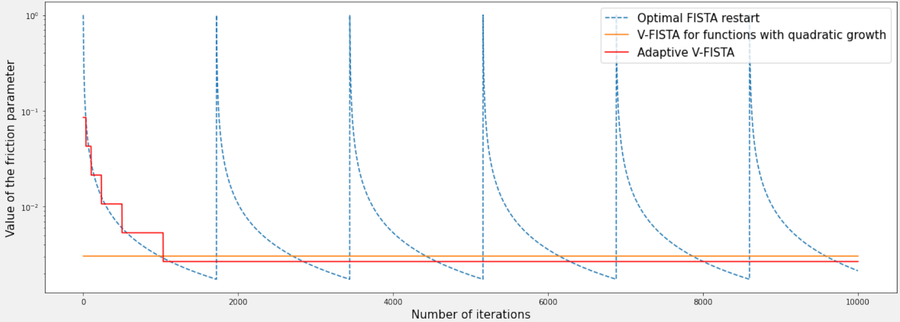


Keep piecewise constant friction to be faster!

### An adaptive procedure for fast methods (L., 2024)

Consider a method $\mathcal{A}$ generating $(x_k)_{k\in\mathbb{N}}$ such that

$$F(x_k) - F^* \leqslant Ae^{-B\sqrt{\frac{\mu}{L}}k}(F(x_0) - F^*)$$

for some $A, B > 0$ **if** $\frac{\mu}{L}$ **is available.**
$\rightarrow$ An adaptive scheme:

- that allows to apply $\mathcal{A}$ when $\frac{\mu}{L}$ is not known with **theoretical guarantees**.
- that can be combined with heuristic techniques (O'Donoghue and Candès, '15) for improved performance.
- which can be extended for methods involving backtracking on $L$ (losing the theoretical guarantees).

Hippolyte Labarrière. Adaptive techniques for linearly fast methods with unknown condition number, currently in writing.

# Conclusion



## Take-away messages

· Inertia is **not impacted** by the non uniqueness of the minimizers.

| | $\mathcal{SC}_\mu$ | $\mathcal{G}^2_\mu$ | $\mathcal{H}^\gamma$ | Convexity |
|---|---|---|---|---|
| Best option | HB | HB | FISTA | FISTA |

· If the condition number is not known $\rightarrow$ FISTA restart... or Adaptive V-FISTA!

## Pending questions:

- Could the Performance Estimation Problem (PEP) approach (Drori and Teboulle,'14, Taylor, Hendrickx and Glineur,'17, Taylor and Drori,'22 ...) allow to find tighter bounds?
- Then, could it help to build faster adaptive schemes?
- Can we obtain better convergence guarantees for adaptive step-size methods (Malitsky and Mishchenko,'20,'24, Barzilai-Borwein stepsize) under growth conditions?

# Conclusion



**Thank you for your attention!**

**Publications and preprints:**

- Jean-François Aujol, Charles Dossal, Hippolyte Labarrière, Aude Rondepierre. FISTA restart using an automatic estimation of the growth parameter, 2021, ⟨hal-03153525v4⟩.
- Jean-François Aujol, Luca Calatroni, Charles Dossal, Hippolyte Labarrière, Aude Rondepierre. Parameter-Free FISTA by Adaptive Restart and Backtracking, 2024, *SIAM Journal on Optimization*.
- Jean-François Aujol, Charles Dossal, Hippolyte Labarrière, Aude Rondepierre. Heavy Ball Momentum for Non-Strongly Convex Optimization, 2024, arXiv preprint arXiv:2403.06930.
- Jean-François Aujol, Charles Dossal, Hippolyte Labarrière, Aude Rondepierre. Strong Convergence of FISTA Iterates under Hölderian and Quadratic Growth Conditions, 2024, arxiv:2407.17063.

**My thesis manuscript (in french!):**

- Hippolyte Labarrière, 2023, Étude de méthodes inertielles en optimisation et leur comportement sous conditions de géométrie.

**Website:**

https://hippolytelbrrr.github.io/

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# The continuous setting

→ **Key tool in convergence analysis**: Link numerical schemes to dynamical systems.

## Gradient descent→ Gradient flow

$$x_k = x_{k-1} - s\nabla F(x_{k-1})$$

$$\iff \frac{x_k - x_{k-1}}{s} = -\nabla F(x_{k-1})$$

# The continuous setting

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$$\downarrow$$

$$\dot{x}(t) + \nabla F(x(t)) = 0.$$

# The continuous setting

**Nesterov's accelerated gradient→Asymptotic vanishing damping system** (Su, Boyd and Candès, '14)

$$\forall k > 0, \begin{cases} x_k = \mathrm{prox}_{sh}\left(y_{k-1} - s\nabla f(y_{k-1})\right), \\ y_k = x_k + \dfrac{k-1}{k+\alpha-1}(x_k - x_{k-1}) \end{cases}$$

$$\downarrow$$

$$\ddot{x}(t) + \frac{\alpha}{t}\dot{x}(t) + \nabla F(x(t)) = 0$$

**Heavy-Ball schemes→ Heavy-Ball Friction system**

$$\forall k > 0, \begin{cases} x_k = \mathrm{prox}_{sh}\left(y_{k-1} - s\nabla f(y_{k-1})\right), \\ y_k = x_k + \alpha(x_k - x_{k-1}), \end{cases}$$

$$\downarrow$$

$$\ddot{x}(t) + \alpha_C\dot{x}(t) + \nabla F(x(t)) = 0$$

# The continuous setting

## Why is this relevant?

- easier computations (derivatives),
- most of the time, convergence properties of the trajectories can be extended to the iterates of the related scheme.

## Back to the discrete setting

Challenging for the following reasons:

- no more derivative,
- several possible discretization choices,
- which condition on the stepsize?

## The continuous setting

Consider the **Heavy-Ball friction system**:

$$\ddot{x}(t) + \alpha \dot{x}(t) + \nabla F(x(t)) = 0$$

Classical Lyapunov energy for this system:

$$\mathcal{E}(t) = F(x(t)) - F^* + \frac{1}{2}\|\lambda(x(t) - x^*) + \dot{x}(t)\|^2$$

# Inertia without uniqueness of the minimizers

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where $x^*(t)$ is the projection of $x(t)$ onto the set of minimizers of $F$ denoted $X^*$.

# Inertia without uniqueness of the minimizers

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where $x^*(t)$ is the projection of $x(t)$ onto the set of minimizers of $F$ denoted $X^*$.

$\rightarrow$ The differentiability of $\mathcal{E}$ depends on the regularity of $X^*$!

If $X^*$ is **sufficiently regular** (e.g. polyhedral), several convergence results can be extended **without the uniqueness assumption** (e.g. Siegel, '19, Aujol, Dossal and Rondepierre, '23).

## An ugly bound

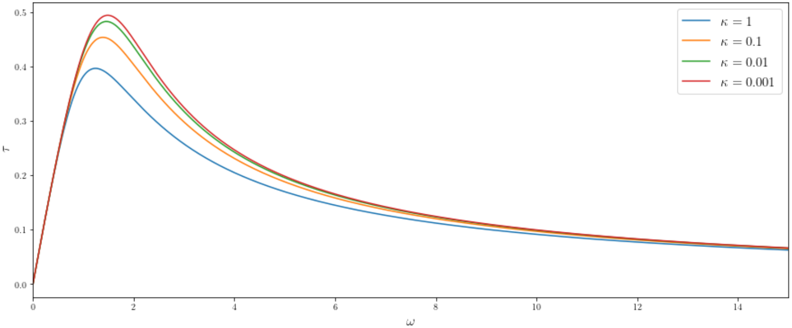
### Main results: V-FISTA

If $F$ satisfies $\mathcal{G}^2_\mu$, $s = \frac{1}{L}$ $\alpha = 1 - \omega\sqrt{\kappa}$ where $\kappa = \frac{\mu}{L}$, $\omega \in \left(0, \frac{1}{\sqrt{\kappa}}\right)$. Then, for any $k \in \mathbb{N}$:

$$F(x_k) - F^* \leqslant \left(1 + (\omega - \tau)^2 + (\omega - \tau)\omega\tau\sqrt{\kappa}\right)\left(1 - \tau\sqrt{\kappa} + \tau^2\kappa\right)^k (F(x_0) - F^*),$$

if

$$\left(1 - \omega\sqrt{\kappa}\right)\tau^3 - \omega\left(2 - \omega\sqrt{\kappa}\right)\tau^2 + (\omega^2 + 2)\tau - \omega \leqslant 0.$$



# An other ugly bound

**Main results: FISTA**

If $F$ satisfies $\mathcal{G}^2_\mu$, $s = \frac{1}{L}$, $\alpha \geqslant 3 + \frac{3}{\sqrt{2}}$, then

$$\forall k \geqslant \frac{3\alpha}{\sqrt{\kappa}},\ F(x_k) - F^* \leqslant \frac{9}{4} e^{-2} M_0 \left(\frac{8e}{3\sqrt{\kappa}}\alpha\right)^{\frac{2\alpha}{3}} k^{-\frac{2\alpha}{3}},$$

where $M_0 = F(x_0) - F^*$ denotes the potential energy of the system at initial time.